Topology Cheatsheet

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October 2, 2025

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1 Topology Cheatsheet

1.1 Berry Curvature Properties

- Zero for single band problems
- Formula

$$\vec{\Omega}^{(n)}(\vec{k}) = i \sum_{m \neq n} \frac{\langle n, \vec{k} | \boldsymbol{\nabla}_{\vec{k}} H | m, \vec{k} \rangle \times \left\langle m, \vec{k} \middle| \boldsymbol{\nabla}_{\vec{k}} H \middle| n, \vec{k} \right\rangle}{(\epsilon_{m,\vec{k}} - \epsilon_{n,\vec{k}})^2} \tag{1}$$

  We can see that for a single band, berry curvature is zero. Also, see that it makes B.C. is a property of the system and not of any single band.

- B.C is largest where bands are closest i.e. near avoided crossings.
- $\sum_m \Omega^{(m)}(\vec{k}) \;=\; 0$ for a given $\vec{k}$.
- Symmetries and $\vec{\Omega}(\vec{k})$
  - Under parity: $\vec{\Omega}_\sigma(\vec{k}) = \vec{\Omega}_\sigma(-\vec{k})$
  - Under TRS: $\vec{\Omega}_\sigma(\vec{k}) = -\vec{\Omega}_{-\sigma}(-\vec{k})$
  - TRS+Parity: $\vec{\Omega}_\sigma(\vec{k}) = \; -\vec{\Omega}_{-\sigma}(\vec{k})$
  - TRS+Parity+ No SOC: $\left| n, \vec{k}, \sigma \right\rangle = \left| n, \vec{k}, -\sigma \right\rangle$ = Spin independent
    $\vec{\Omega}(\vec{k}) = \vec{\Omega}(-\vec{k}) = -\vec{\Omega}(-\vec{k}) = 0$, so no berry phase (probably the reason why conventional solid state textbooks get away by removing this physics altogether)
  - TRS+Parity+Strong SOC (e.g. heavy metals): $\left| n, \vec{k}, \sigma \right\rangle \neq \left| n, \vec{k}, -\sigma \right\rangle =$, so $\Omega \neq 0$.
    However, $\vec{\Omega}_\sigma(\vec{k}) = \; -\vec{\Omega}_{-\sigma}(\vec{k})$, so opposite spins have **Opposite anomalous velocities**.

- Physical realisations of 3 cases
  - Broken Inversion: MoS_2 breaks inversion symmetry in a graphene type lattice, possess opposite BC for two valleys.
    As a result, when an electric field is applied, electrons in different valley deflect differently, giving a transverse hall type voltage (even in absence of $\vec{B}$). This is called **Valley Hall effect**.
    Here, $\vec{\Omega}(\vec{k})$ = -$\vec{\Omega}(-\vec{k})$ (spinless model), and $h(-K) = h(K')$, hence berry curvature are of opposite signs.
  - Broken Time reversal: bands in magnetic metals like Fe, Co, Ni, and their alloys can have non-zero Berry curvatures.
    Leads to **Anomalous Hall Effect**: a voltage $V_\perp \perp$ to $\vec{\mu}, \vec{E}$ that can be much larger than the ordinary V_{hall} produced by an $\vec{B}_{applied}$.
    Why is there a spin current in anomalous hall effect?: Because time reversal symmetry is broken (or eqv you've unequal numbers of spin $\uparrow, \downarrow$ giving you the magnet), there is a non-zero spin current.
  - Strong SOC+Parity+TRS: Heavy metals like Pt and Ta.
    Apply $\vec{E} \neq 0 \implies$ Opposite spins get deflected to opposite sides ($\vec{\Omega}_\sigma(\vec{k}) = -\vec{\Omega}_{-\sigma}(\vec{k})$). Due to TRS, $n_\uparrow = n_\downarrow \implies$ No net Hall voltage. However, $j_{spin} \neq 0 \implies$ **Spin-Hall Effect**

1.2 Why an Insulator?

Because when we define berry phase, we impose adiabatic theorem and demand that a given state $|n\rangle$ doesn't cross another state $|m\rangle$. If it did, we get a level crossing, after which when the states separate, it's not possible to conclude which superposition of the states the system now lives in.
Moreover, to get Topological invariants, we often integrate over filled bands, hence partially filled bands must be empty.

1.3 Topological Invariants: Broad

- Why does an Insulator even have a finite "conductivity"?
  Insulator - Filled bands - Hence it can't generate voltage/transport current parallel to electric field.
  However, there can still be a nonzero Hall conductivity associated with charge currents and electric fields that are perpendicular (e.g., the band can support a flowing charge current that generates a transverse voltage, but no longitudinal voltage)
- A non-zero Chern number equals to some non-zero hall conductivity - Why?
  Because hall conductivity is obtained by integrating the transverse anomalous velocity $\vec{\Omega}(\vec{k})$ over all of the states in the filled bands.
  Non-zero C $\implies$ Quantized Hall conductance
  Can understand: Q.Hall (TRS breaking $\implies C \neq 0$), QAHE ($\vec{B} = 0$ but TRS is still broken).
  How to Physically understand the "transverse" velocity in Q.Hall effect?"
- Topological represent forms of matter different from their non-topological counterparts. For e.g., at the edge of materials with different topological invariants, the bandgap must go to 0 to allow for an edge state.

1.4 SSH Model

$$H_{SSH} = (t + \delta t + (t - \delta t)cosk)\sigma_x + (t - \delta t)sink\ \sigma_y \tag{2}$$

3 perspectives

- Winding Number - Berry Phase perspective
  Why Topological? In the d_x, d_y plane, you can make variety of curves that won't (consistently) have origin in them and all of them are guranteed to have the same winding number (hence same topological properties).

- Mapping onto a 1d Dirac equation

$$\mathcal{H} = (t-\delta t)(sink)\sigma_x + (t+\delta t + (t-\delta t)cosk)\sigma_z \tag{3}$$

$$H_{eff} = (t-\delta t)(-k)\sigma_x + (2\delta t + (t-\delta t)\frac{k^2}{2})\sigma_z \tag{4}$$

  Expand around $k \to k+\pi$. Given the SSH $\mathcal{H}$, we can do a unitary transformation (workout exactly how) sending H_{ssh} it to 1d dirac hamiltonian. This gurantees edge modes for $sgn(\delta t(\delta t - t)) > 0 \implies \delta t < 0$ (corresp to $mB > 0$).
  Edge mode localization : $\delta t \downarrow, \xi \uparrow$

- Level crossing, Topological Phase transition and berry curvature transfer:
  As bandgap decreases, the integrated berry phase (over some small batch of the 1d BZ) is $\mp\pi/2$ near $k = 0$ for the GS/ES, while it increasingly becomes $\pm\pi/2$ at $k = \pm\pi/a$ (this is for trivial case i.e. $\delta t > 0$).
  Bands touch $\implies$ Berry phase gets transferred $\implies$
  GS has $-\pi/2$ (band minima) + $-\pi/2$ (band extrema, now reversed) $\implies$ net BP = π.
  Lesson: Band touchings/level crossings transfer berry phase, turning trivial into topological.

1.5 Level Crossings in a general Topological System

- 2D Graphene type lattice:

$$H_{eff} = \begin{bmatrix} m & U(k_x \mp ik_y) \\ U(k_x \pm ik_y) & -m \end{bmatrix} = \vec{d}(\vec{k}) \cdot \vec{\sigma} \tag{5}$$

  (± - @K,K').
  Similarity to SSH = near the minima, one component if k independent (i.e.m), others are linearly dependent. BP = $\pm\pi sgn(m) \implies$ as you tune m from -ve to +ve (by chaning the order hierarchy of ϵ_A, ϵ_b, onsite energies), we flip the BP of both GS (at K, K') by 2π. However, since they're still equal and opposite, the net integrated BC is 0.
  Break TRS $\implies$ one Valley flips m while another doesn't
  $\implies$ C = $\frac{1}{2\pi}\int \Omega(\vec{k})d^2\vec{k}$ goes from 0 to $\pm 2\pi$ (**How exactly??**) $\implies$ we get QAHE

- A similar reasoning works in 3d TI too, but with some caveats. See @Dan ralph here.

1.6 Quantum Hall effect

Classical Expectations

- Why measure ρ_{xy}?: independent of scattering time (τ), hence showcases property inherent to the 2d system
  Also, $R_{xy} = \rho_{xy}$ (surprisingly), so what you measure is exactly what you want.

- Drude predictions: $\rho_{xx} = \dfrac{m}{ne^2\tau}$, $\rho_{xy} = \dfrac{B}{ne}$

- IQHE:
  - Plateaued Values: $\rho_{xy} = \dfrac{2\pi\hbar}{e^2}\dfrac{1}{\nu}$, over a range of magentic fields.
  - Centre of each Plateau: $B = \dfrac{2\pi n\hbar}{e}\dfrac{1}{\nu} = \dfrac{n\Phi_0}{\nu}$
  - $\rho_{xx} = 0$ for every Plateau, spiking to a large value in between two of them.

- What is IQHE: Perfect conductor or Perfect insulators?
  - Here $\rho_{xx} = 0$, which leads to $\sigma_{xx} = 0$ too (provided $\rho_{xy} \neq 0$)
  - But usually, $\rho_{xx} = 0 \implies$ perfect conductor, $\sigma_{xx} = 0 \implies$ perfect insulator
  - Solution: This is a artifact of $\tau \to \infty$ limit of Drude Model. Here, $\vec{J} \perp \vec{E}$, so $\vec{E}\cdot\vec{J} = 0$ $\vec{E}\cdot\vec{J}$ = work done in accelerating charges

- $\rho_{xx} = 0$: No energy gets dissipated, $\sigma_{xx} = 0$: No current in longitudinal direction.
- In the $\tau \to \infty$ limit, we don't get a longitudinal current in equilibrium ($\frac{d\vec{v}}{dt} = 0$), only transverse for E_x.

1.7 Jackiw-Rebbi Solution

1.8 Anti Unitary Symmetries

Hand written notes