Bosonization

… and some Applications

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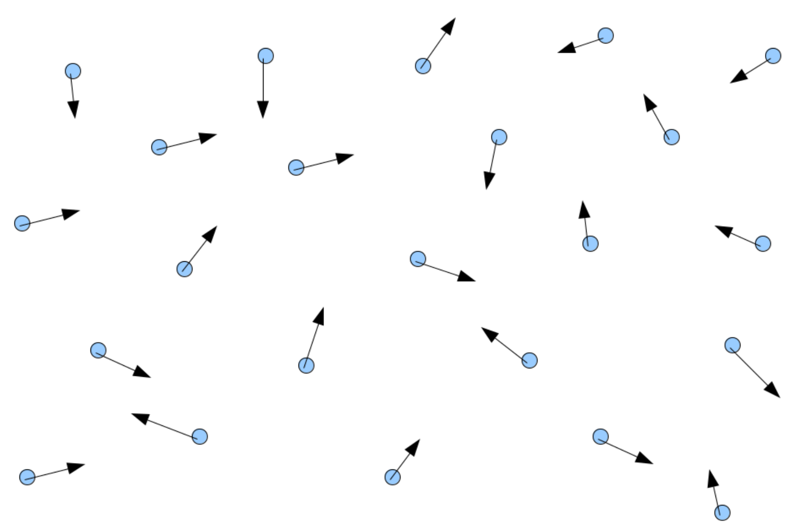
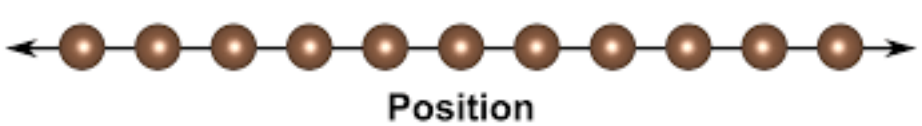
Agenda

- Bosonization: Recap
- Example : 4 fermion interactions
- Some non-trivial applications:
    - 1: Calculating conductance of an 1d interacting wire
    - 2: Calculating the conductance in presence of isolated and impurity

Recap

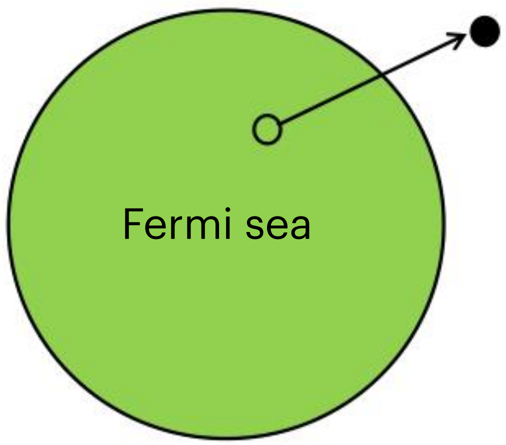
- Peculiarity of 1d: No individual excitations can work ("Collectivisation" happens)





- Excitations can have fundamentally different nature.

- Crucial excitations of e-gas: Particle-Hole excitations



Destroy a particle at $\overrightarrow{k}$ and create one at $\overrightarrow{k} + \overrightarrow{q}$ implies it has a fixed and well-defined momenta: $\overrightarrow{p} = \hbar \overrightarrow{q}$

- It's energy: $E_k(q) := \xi(\overrightarrow{k} + \overrightarrow{q}) - \xi(\overrightarrow{k}) = q^2 + c\,\overrightarrow{k} \cdot \overrightarrow{q} \implies$ k dep., not well defined!

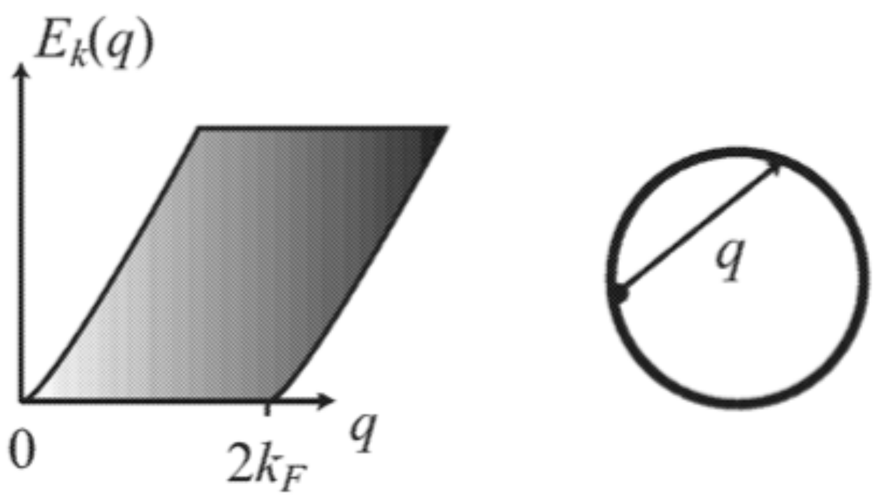


Fig. Particle-hole spectrum in 2d (or higher)

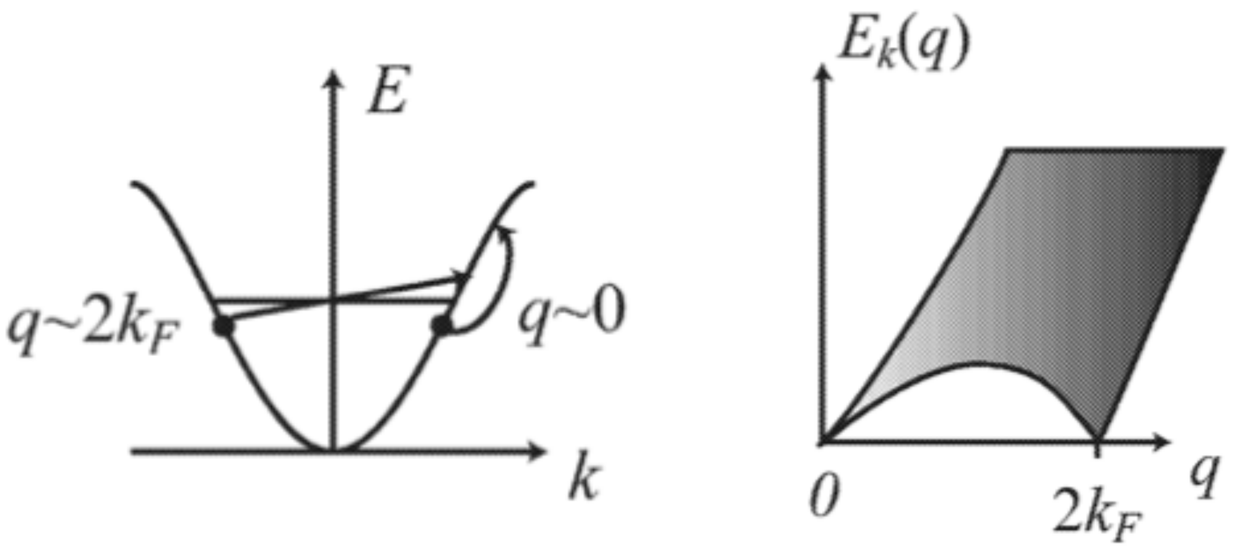


Fig. Particle-hole spectrum in 1d

- Focus near $q = 0$ part :

$$< E_k(q) >_{k \, \epsilon \, [k_F - q, \, k_F]} = \frac{k_F q}{m} \quad \& \quad \delta E(q) = max(E_k(q)) - min(E_k(q)) = \frac{q^2}{m}$$

- We make two observations:

1. Avg energy is k independent

2. δE goes to zero much faster than $< E >$

- Meaning: P-h excitations are "**<u>well defined</u>**" excitations (well defined energy and momentum) which become longer and longer lived as when $q \to 0$.

- One then translates an original fermionic problem in terms of these "excitations" (bosonic) $\to$ premise of bosonization.

$$b_q^\dagger \sim c_{k+q}^\dagger c_k \to \sum_k c_{k+q}^\dagger c_k \qquad \rho(x) = \psi^\dagger(x)\psi(x) \to \rho(q) \propto \sum_k c_{k+q}^\dagger c_k \sim b_q^\dagger$$

Example application

- For simplicity, let's work with a free-electron dispersion: $\epsilon(k) = v_F k$

$$H_0 = v_F \sum_{k=-\infty}^{\infty} k \ : c^\dagger_{R,k} c_{R,k} :$$

- How does it look in bosonic language?

$$\begin{aligned} b^\dagger_{R,q} &= \frac{1}{\sqrt{n_q}} \sum_{k=-\infty}^{\infty} c^\dagger_{R,k+q} c_{R,k} \ , \\ b_{R,q} &= \frac{1}{\sqrt{n_q}} \sum_{k=-\infty}^{\infty} c^\dagger_{R,k-q} c_{R,k} \ , \\ q &= \frac{2\pi}{L} n_q \ , \end{aligned} \longrightarrow \quad H_0 = v_F \sum_{q>0} q b^\dagger_{R,q} b_{R,q}$$

Free fermions → Free bosons

- Now put in interactions :

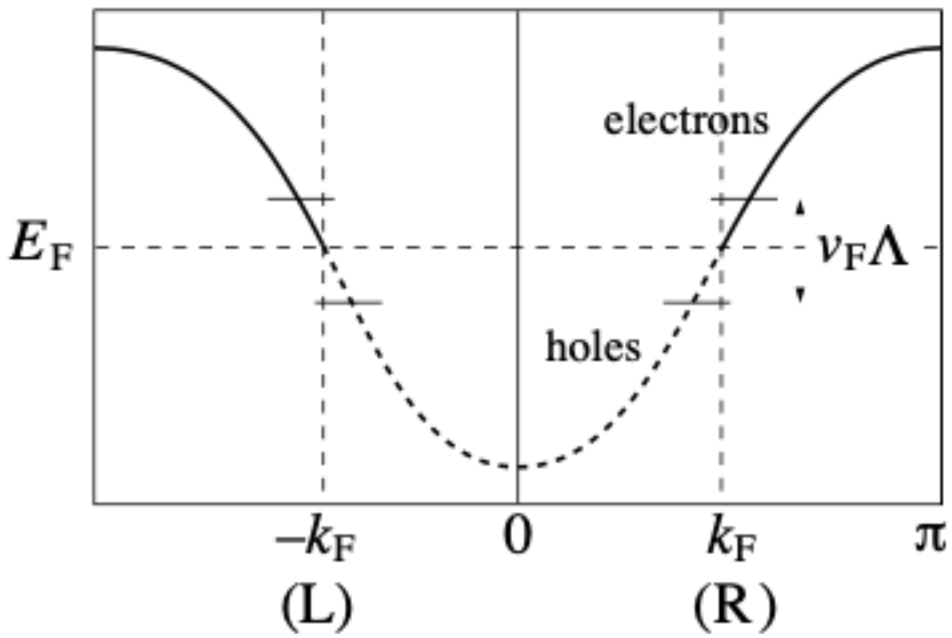
$$V = \frac{1}{2} \int_0^L g_4 \ {\rho_R}^2(x) \longrightarrow \frac{g_4}{2\pi} \sum_{q>0} q \ b^\dagger_{R,q} b_{R,q}$$

Still Quadratic - Exactly soluble!

$$v_F \to v_F + \frac{g_4}{2\pi}$$

- Stretch: same dispersion + 2 species of fermions

$$H_0 \;=\; v_F \sum_{k=-\infty}^{\infty} k \,[\; : c^\dagger_{R,k} c_{R,k} \;+\; c^\dagger_{L,k} c_{L,k} \; : \;]$$



$$H_0 \;=\; v_F \sum_{q>0} q \;(\; b^\dagger_{R,q} b_{R,q} \;+\; b^\dagger_{L,q} b_{L,q} \;)$$

Add interactions

$$V \;=\; \frac{1}{2}\int_0^L dx \;[\; 2g_2\, \rho_R(x)\rho_L(x) \;+\; g_4\;(\; \rho_R^2(x) \;+\; \rho_L^2(x)\;)\;]$$

Possibly Backscattering term   Forward scattering term

- Bosonic rewrite? -

$$H = \sum_{q>0} q\;[v_F\;(b^\dagger_{R,q} b_{R,q} + b^\dagger_{L,q} b_{L,q}) \;+\; \frac{g_2}{2\pi}(b^\dagger_{R,q} b^\dagger_{L,q} + b_{R,q} b_{L,q}) \;+\; \frac{g_4}{2\pi}(b^\dagger_{R,q} b_{R,q} + b^\dagger_{L,q} b_{L,q})]$$

Still Quadratic - ~~Exactly soluble! -~~ Bogoliubov transf ✓

- Define two parameters:

$$
\begin{aligned}
v &= \left[\left(v_F + \frac{g_4}{2\pi} - \frac{g_2}{2\pi} \right) \left(v_F + \frac{g_4}{2\pi} + \frac{g_2}{2\pi} \right) \right]^{1/2} , \\
K &= \left[\left(v_F + \frac{g_4}{2\pi} - \frac{g_2}{2\pi} \right) / \left(v_F + \frac{g_4}{2\pi} + \frac{g_2}{2\pi} \right) \right]^{1/2}
\end{aligned}
$$

if g_2>0, K<1 (repulsive attraction) , K=1 for non-interacting.
K, v characterise such a system (and often are measurable - see later).

$$
\begin{aligned}
H = & \sum_{q>0} vq \left[\tilde{b}^\dagger_{R,q} \tilde{b}_{R,q} + \tilde{b}^\dagger_{L,q} \tilde{b}_{L,q} \right] \\
& + \frac{\pi v}{2L} \left[\frac{1}{K} (\hat{N}_R + \hat{N}_L)^2 + K (\hat{N}_R - \hat{N}_L)^2 \right]
\end{aligned}
$$

- Define: $b_q \to \phi(x) \sim \sum_q e^{iqx} b_q$, then more transparently

$$\boxed{\mathcal{L} = \frac{1}{2v_F} (\partial_t \phi)^2 - \frac{v_F}{2} (\partial_x \phi)^2} \longrightarrow \boxed{\mathcal{L} = \frac{1}{2vK} (\partial_t \phi)^2 - \frac{v}{2K} (\partial_x \phi)^2}$$

Non-interacting Interacting

Applications

Conductance of 1d wires

- Usual approach computing conductivity (with results such as $\frac{2e^2}{\hbar}$) needs to be modified as interacting system renders such ideas moot.

- Final Result:

  While the wire without impurity gives an interesting answer for conductivity, a wire with isolated impurity displays different physics altogether (interactions).

- From standard QM, we know for usual 1d wires, $\exists$ transmission and reflection amplitude from a scattering site. But for an interacting wire, as we'll see, no matter how small the scattering potential is, depending on the **sign** e-e interactions, either T=1 (i.e.full transmission) or T=0 (full reflection).

Conductance in presence of impurity

- Start with the imaginary time action:

$$S_E = \frac{1}{2K}\int d\tau \int dx \; [\frac{1}{v}(\partial_\tau \phi)^2 + v(\partial_x \phi)^2] \; .$$

- Add a term an interaction term of the form

$$S_{\rm int} = \int dx d\tau \; V(x) \psi^\dagger(x) \psi(x) \; .$$

where V(x) is weak and centred at origin viz. $V(x) = \lambda\delta(x)$ for small λ

- 1d QM result (non-interacting)

$$R = \frac{\lambda^2}{\lambda^2 + k^2} \quad \text{and} \quad T = \frac{k^2}{\lambda^2 + k^2}$$

non-zero conductance irrespective of size of λ.

- After a little algebra, one ends up with an action

$$\boxed{S = S_E + S_{\rm int} = S_E - \frac{\lambda}{\sqrt{\pi}}\partial_x \phi(0) + \frac{\lambda}{2\pi\alpha}\int d\tau \cos 2\sqrt{\pi}[\phi_R(0) + \phi_L(0)]}$$

- One can now compute the conductivity using Kubo formula as

$$I(x) = \int_0^L dx' \int \frac{d\omega}{2\pi}\ e^{-i\omega t}\sigma_\omega(x,x')E_\omega(x')$$

$$\sigma_\omega(x,x') = -\frac{e^2}{\bar{\omega}}\int_0^\beta\ d\tau\ \ < T_\tau j(x,\tau)j(x',0) > e^{-i\bar{\omega}\tau}$$

- However, one takes another approach: namely, we want to compute the conductance through this barrier at low energies. One way to do that is to see whether this barrier coupling strength grows or becomes smaller as we go to lower energy scales. To check that, performs steps of a renormalization group analysis.

- Why low energy? - Different models have different values for λ, however they may share the same qualitative properties. So reduce energy scales till RG flow stops and we end up at the fixed pnt $\mathscr{H}$.

RG analysis

- Since perturbation is fixed in space, integrate all variables away from origin and write down action in terms of $\phi(x=0,\tau)$.

$$S = S_E + S_{\text{int}} \xrightarrow[\int_{-\infty}^{0} dx + \int_{0}^{\infty} dx]{} S_{\text{eff}} = \frac{1}{K}\int \frac{d\bar{\omega}}{2\pi} |\bar{\omega}|\phi^2$$

$$\downarrow$$

$$\boxed{S = \frac{1}{2K}\int \frac{d\bar{\omega}}{2\pi} |\bar{\omega}|\phi(\bar{\omega})^2 + \lambda \int \frac{d\bar{\omega}}{2\pi} \cos[2\sqrt{\pi}\phi(\bar{\omega})]}$$

- 3 step RG procedure:

Choose a high frequency cutoff Λ and scale $\Lambda \to \Lambda/s, \;\; s > 1$

$\rightarrow$

Divide $\phi(\omega)$ into $\phi_>(\omega)$ (fast modes) and $\phi_<(\omega)$ (slow modes) for $\omega \gtrless \Lambda/s$

$\rightarrow$

Integrate out $\phi_>(\omega)$ and rescale $\omega \to s\cdot\omega$ for fair comparison and see how Interactions change

- To the lowest order, one finds that

$$\lambda \int d\tau \cos 2\sqrt{\pi}\phi_<(x=0,\tau) \rightarrow \lambda s^{1-d} \int d\tau \cos 2\sqrt{\pi}\phi_<(x=0,\tau)$$

where d is the dimension of the cosine operator, which is **K** for this case.

- Therefore, we can further conclude

$$\begin{aligned} & s = 1 + dl, \\ & \lambda' = \lambda(1+dl)^{1-K} \\ \Rightarrow \quad & \lambda' - \lambda = (1-K)\lambda dl \\ \Rightarrow \quad & \frac{d\lambda}{dl} = (1-K)\lambda\ . \end{aligned}$$

- As l increases (i.e. RG flow occurs) for K>1 (i.e. g_2>0 & repulsive), barrier strength vanishes. While for K<1 (i.e. g_2<0 & attractive), barrier strength grows. This leads to healing and cutoff of wire resp.

- Also, for K=1, coupling is marginal, which is to be expected.
- Thus the wire “heals” i.e. T=1 or “cuts” i.e. T=0 iff attraction is attractive/repulsive respectively.
- Similar conclusions are also drawn at strong barrier strength analysis.

Thank you for your patience!