# Physics 211C: Solid State Physics

Instructor: Prof. Tarun Grover

by Guru Kalyan Jayasingh

## Topic 3: Quantum Magnetism

* LSM theorem: introduction & proof
* Oshikawa's flux threading argument
* Spin representations
* Large S approach & spin waves
* Large N approach to spin systems
* Practice problems

# Quantum Magnetism

- gapless phases
  - Goldstone modes
  - Gapless fractionalized excitations
    - e.g. 1 :- spinons
- Gapped
  - Top. protected phases (e.g. Anyons)
  - SPT

e.g. 2 :- Heisenberg chain in 1d

$+|J| \sum_r \vec{S}_r \cdot \vec{S}_{r+1}$ → solved by Bethe (1931)

⇒ has gapless fractionalized excitations

Simplest model of a QSL

QSL phase := no local OP, $\Psi_{GS}$ can't be obtained from a product state via a short depth unitary quantum circuit.

⇒ Low energy theory of heisenberg magnet is just spin-1 spinons coupled to a U(1) gauge field.

( $S = f^{\dagger} \sigma f$ → f coupled to a U(1) gauge field

→ Basically gives correct answer at lowest order

→ apply $\frac{1}{N}$ expansion, make f N comp. fermion)

⇒ Was among the 1st uses of LSM

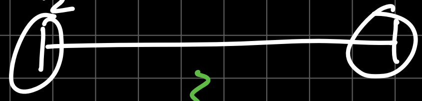
## LSM theorem (for heisenberg chain)

spin $\frac{1}{2}$ vs Spin 1

gapless (e.g. spinons) — gapped (SPT phase)

$\Downarrow$

spin $\frac{1}{2}$ objects at ends of a spin 1 system (for open boundary)



SPT phase in 1d

↳ boundaries are either gapless/fractionalized

# LSM says $s=\frac{1}{2}$ is gapless but doesn't say alot about $s=1$.

## Broad idea of LSM

We want to talk about spin-s chains & investigate low lying spectrum.

e.g. Spin-s Heisenberg AFM chain $d=1$

$\Downarrow$

what happens in 1d?

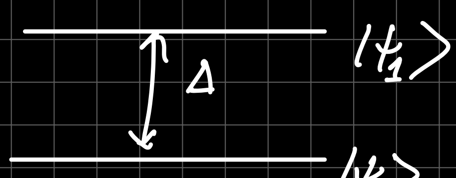
One possibility:- $O(N)$ NLSM $d=1$

$\Downarrow$

can show that $T=0$ it's gapped,

"Exp. decaying correlation functions"

$\Rightarrow$ Classical result, motivates that its very hard to get gapless stuff in 1d (i.e. Haldane's result is surprising)

$\Rightarrow$ 


$\Delta \neq 0$ as $L \to \infty$

reflects a most reasonable intuition for 1d system (w/o knowledge of QM, bethe ansatz, boson. etcetc.)

$\Rightarrow$ rotor model in 1d: $\sum_i \frac{n_i^2}{2I} + \sum \cos(\theta_i - \theta_{i+1}) \equiv O(2)$ model

$$[n, \theta] = i$$

Using Quantum O(N) model in 1d, one can show

$$e^{-\frac{1}{g^2}\int (\partial_\mu n)^2 \, dx \, d\tau}$$

$$\frac{d(g^2)}{dl} = +g^2 (N-2) \rightarrow \text{keeps growing}$$

$N=2$ (KT physics)

$\hookrightarrow g \approx T$

$\hookrightarrow g \to \infty$, system is completely disordered at $T=0$

**LSM result:-** When spin $s$ is $\frac{1}{2}$ odd integer, one can construct a low lying state with energy $E_1$ s.t.

$$E_1 - E_0 \lesssim \frac{1}{L}$$

**Argument:-** $T_x \underbrace{|\Psi_0\rangle}_{\text{GS}} = e^{ik_0} |\Psi_0\rangle$ [Assume translational invariance]

transl. operator

$$|\Psi_1\rangle = \mathcal{U} |\Psi_0\rangle = e^{\frac{2\pi i}{L} \sum_{x=1}^{L} x S^z(x)} |\Psi_0\rangle$$

work in PBC i.e. $S_{L+1} = S_1$

now we can show

① $T_x |\Psi_1\rangle = e^{ik_0} e^{i\pi} |\Psi_1\rangle$

② $\langle \Psi_1 | H | \Psi_1 \rangle = E_1 \quad E_1 - E_0 < \frac{1}{L}$

$\Bigg\}$ for spin $\frac{1}{2}$

$\therefore \nexists$ a gapped GS for spin $\frac{1}{2}$ chain

$$T_x|\psi_1\rangle = T_x\, \mathcal{U}|\psi_0\rangle = T_x \mathcal{U} T_x^+ T_x |\psi_0\rangle$$

$$T_x S^z_x T_x^+ = S^z_{x+1}$$

$$\therefore \quad T_x \mathcal{U} T_x^+ \rightarrow \exp\left(\frac{2\pi i}{L}\sum x S^z_{x+1}\right) = \mathcal{U}\exp\left(-\frac{2\pi i}{L}\sum_{x=1}^{L} S^z_x\right)$$

$$1\cdot S^2 + 2\cdot S^3 - - - + L\, S^{L+1}$$

$$\exp\Big(\underbrace{\frac{2\pi i}{L}\cdot L\cdot S^z_1}_{\text{Boundary term}}\Big)$$

$$L S^L \qquad S^1 \qquad L S^L$$

$$(L-1) \qquad (L S^1) \qquad (L-1)\, S^{L-1}$$

$$= \mathcal{U}\exp\left(-\frac{2\pi i}{L}\sum_{x=1}^{L} S^z_x\right)\exp\left(2\pi i S^z_1\right)$$

$$\Downarrow$$

$\sum_{x=1} S^z(x) = S^z_{total} = 0$ in true for ground state (Auerbach #5.1) } Marshall's sign rule

"AFM on a bipartite lattice, GS is singlet i.e. $S^z_{net} = 0$"
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\ \ \|$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \ \sum_j S^z_j$

$$= \mathcal{U}\exp\left(2\pi i S^z_1\right) e^{ik_0}|\psi_0\rangle$$

$\exp(2\pi i S^z_1) \to$ crs $= \pm\frac{1}{2}$ (general identity being

$$\exp(i2\pi S^z_1) = \begin{cases} 1, & S = 0, 1, 2, \ldots \\ -1, & S = \frac{1}{2}, \frac{3}{2}, \ldots \end{cases}\Bigg)$$

$$= e^{i(k_0+\pi)}\,\mathcal{U}|\psi_0\rangle$$

$$\therefore \langle\psi_1|\psi_0\rangle = 0$$

now

$$\mathcal{H} = \sum S^z_x S^z_{x+1} + \frac{1}{2}\sum_x S^+_x S^-_{x+1} + S^+_{x+1}$$

$$S^+_x = S^x_x + i S^y_x \qquad S^-_x = S^x_x - i S^y_x$$

$$E_1 = \langle \psi_0 | \underbrace{U^\dagger H U}_{} | \psi_0 \rangle$$

$$\sum_x S^z_x S^z_{x+1} + \left(\frac{1}{2}\right)\left[\,\cdots\cdots\,\right]$$

now $e^{-i\alpha S^z} S^+ e^{i\alpha S^z}$

$$= 1 - i\alpha S^+ + \frac{(i\alpha)^2}{2} S^+ \cdots$$

$$\left( \frac{BCH}{e^A B e^{-A}} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{n!}[A,[A\cdots[A,B]]\cdots \right)$$

$$= e^{-i\alpha} S^+$$

$$\therefore\ U^\dagger S^+_x S^-_{x+1} U$$

$$[S^z, S^+] = S^+$$

$$[S^z, S^-] = -S^-$$

$$= U^\dagger S^+_x U\, U^\dagger S^-_{x+1} U$$

$$= e^{-\frac{2\pi i x}{L}} S^+_x\, e^{+\frac{2\pi i}{L}(x+1)} S^-_{x+1}$$

$$= S^+_x S^-_{x+1}\, e^{\frac{2\pi i}{L}}$$

$$\therefore\ U^\dagger H U = \sum S^z_x S^z_{x+1} + \left(\frac{1}{2}\right) e^{\frac{2\pi i}{L}} \sum S^+_x S^-_{x+1} + \frac{1}{2} e^{-\frac{2\pi i}{L}} \sum S^-_x S^+_{x+1}$$

$$\langle \psi_0 | H | \psi_0 \rangle = \langle \psi_0 | S^z_x S^z_{x+1} | \psi_0 \rangle + \frac{1}{2}\left(\cos\frac{2\pi}{L} - 1\right) \sum \langle \psi_0 | S^+_x S^-_{x+1} + S^-_x S^+_{x+1} | \psi_0 \rangle$$

$$+ \frac{i}{2}\left(\sin\frac{2\pi}{L}\right) \sum_x \langle \psi_0 | S^+_x S^-_{x+1} - S^-_x S^+_{x+1} | \psi_0 \rangle$$

$$\parallel$$

$$\underbrace{S^x_i S^y_j - S^y_i S^x_j}$$

because of TRS $0 =$

TRS: define antiunitary by

$$\left.\begin{aligned} S^x &\to S^y \equiv O_x \\ S^y &\to S^x \equiv O_y \\ S^z &\to S^z \equiv O_z \\ i &\to -i \equiv c \end{aligned}\right\}$$ $H_{heisenberg}$ is invariant under this

Check:-
$$[S^y, S^x] = -i\, S^z = (-i)\, S^z$$
$$[O^x, O^y] = c\, O^z \quad \text{\& similarly}$$

$\therefore$ 2nd term vanishes

$$\therefore \quad E_1 \lesssim E_0 + \frac{1}{L^2} \cdot \underbrace{S^2}_{\text{bounded (local operator)}} \cdot L \sim E_0 + \frac{1}{L}$$

in d-dim. $\quad \dfrac{L^D}{L^2} \sim L^{D-2}$ (in $D \geq 2$, inconclusive)

$\Downarrow$

where Oshikawa's argument helps here

$\therefore$ By using TRS & translational inv.,
odd spin Heisenberg model is gapless for $d=1$.

# Oshikawa's argument for higher dimensions

**Statement:-** Consider a system of bosons / fermions at filling $\nu = \frac{p}{q}$, $\exists$ atleast $q$ low lying states. (including GS)

(one can consider $s=\frac{1}{2}$ $\mathcal{H}_{heisenberg}$ as bosons at $\frac{1}{2}$ filling. How?)

e.g.

$\xrightarrow{T_1}$

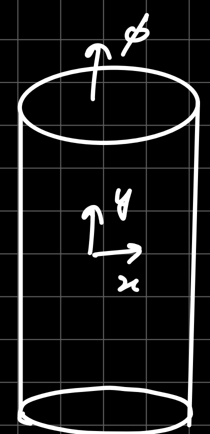
$\frac{1}{2}$ filling GS under repulsion → 2 degenerate GS

\# Let's take a specific hamiltonian

$$\mathcal{H} = \mathcal{H}_{hopping} + \mathcal{H}_{int}$$

$$= -t \sum (b_i^{\dagger} b_j + h.c.) + V(\{n_i\})$$

$d = 2$




$L_x \times L_y$

idea:- put system on a cylinder

↓

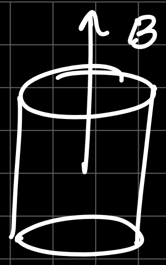
$b \to b e^{i\theta}$ has conserved $U(1)$ global charge

↓

introduce background gauge field through this cylinder & thread a flux through the cylinder

⇙

system acquires a momentum as $\Phi_B : 0 \to 2\pi$, for $q \neq 1$ → new state obtained is still low lying & orthogonal




$$\frac{\partial B}{\partial t} \sim \nabla\times\vec{E} \Rightarrow \int \vec{E}\cdot d\vec{S} = \oint \vec{E}\cdot d\vec{l} = \frac{\partial \phi}{\partial t}$$

$\vec{E}$ changes momentum

$$\Delta p \sim q\int E\,dt \sim \Delta\phi$$

(Does it work only for charged bosons?)

$$\mathcal{H}_0 = b_r^\dagger b_{r'} + h.c. \xrightarrow[\text{minimal coupling}]{B\neq 0} \left(b_r^\dagger b_{r'}\, e^{i a_{rr'}} + h.c.\right)$$

may/may not

for this argument, we give a "fictitious" charge e coupled to a "fictitious" EM field.

$$a_{rr'} = \frac{q}{\hbar}\int \vec{A}\cdot\vec{e}_{rr'}$$

Choose $a$ to be constant in space for threaded flux

$$a_{r,r+\hat{x}} = \frac{\phi}{L_x}, \quad a_{r,r+\hat{y}} = 0$$

$$\therefore \mathcal{H}(\phi) = -t\sum_r \left(b_r^\dagger b_{r+x}\, e^{+i\frac{\Phi}{L_x}} + h.c.\right)$$

$$-t\sum_r \left(b_r^\dagger b_{r+y}\, e^{i\times 0} + h.c.\right)$$

$$+\, V(\{n_r\})$$

$\hookrightarrow$ gauge invariant

Increase $\Phi : 0 \to 2\pi$ adiabatically (idea is that $\mathcal{H}(\Phi=0)$ & $\mathcal{H}(\Phi=2\pi)$ are same)

$$U = \exp\left(\frac{2\pi i}{L_x}\sum_{\vec{r}} x\, n_{\vec{r}}\right)$$

<u>Claim</u>: $\mathcal{U}^{+} \mathcal{H}(\Phi = 2\pi)\, \mathcal{U} = \mathcal{H}(\Phi = 0)$ (i.e. they're the same evalue)

$[n_\gamma, b_\gamma] = b_\gamma$

$[n_\gamma, b_\gamma^+] = -b_\gamma^+$

$\mathcal{U}^+ \; b_\gamma^+ \; \mathcal{U}$

$= \exp\left(-\frac{2\pi i}{L_x}(-x_\gamma)\right) b_\gamma^+ \quad$ (BCH)

$= \exp\left(\frac{2\pi i}{L_x} x\right) b_\gamma^+$

$\mathcal{U}^+ b_{\gamma+\hat{x}} \mathcal{U} = \exp\left(-\frac{2\pi i}{L_x}(x+1)\right) b_{\gamma+\hat{x}}$

$\rightarrow b_\gamma^+ b_{\gamma+x} \exp\left(-\frac{2\pi i}{L_x}\right)$

cancels out

$e^{i a_{\gamma\gamma'}}$

Let $\mathcal{H}(\phi=0)|\psi_0\rangle = E_0|\psi_0\rangle$

$|\psi_0\rangle \xrightarrow[\phi:0\to 2\pi]{} |\psi_0'\rangle$ has same crystal momentum as $\psi_0$

( $\because [\mathcal{H}(\phi), T_x] = 0$ i.e. $k$ doesn't change once labelled)

because gauge is chosen as such

$T_x e^{-i\int \mathcal{H}(t)\, dt} |\psi_0\rangle$

$= e^{ik_0} e^{-i\int \mathcal{H}_t dt} |\psi_0\rangle$

$\langle \psi_0' | \mathcal{U}^+ \mathcal{H}(\phi = 0)\, \mathcal{U} |\psi_0'\rangle = \langle \psi_0' | \mathcal{H}(\phi = 2\pi) |\psi_0'\rangle$

since done adiabatically.

$\simeq \langle \psi_0 | \mathcal{H}(\phi=0) | \psi_0\rangle$

i.e. $|\psi_0\rangle \rightarrow |\psi_0'\rangle$ adiabatically.

$= E_0$

certainly true if system has a gap.
(search for contradict$^n$)

$\therefore$ For an apples to apples comparison, we compare $\mathcal{U}|\psi_0'\rangle$ with $|\psi_0\rangle$.

$$T_x \mathcal{U}|\psi_0'\rangle = \underline{\underline{T_x \mathcal{U} T_x^{\dagger}}}\, T_x|\psi_0'\rangle$$

LSM like

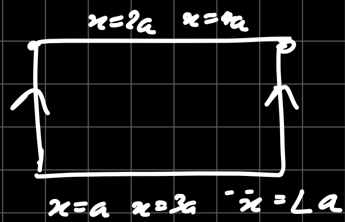
$$= \mathcal{U}\exp\left(\frac{2\pi i}{L_x}\sum_{\vec r} n_{\vec r}\right)\exp\left(-\frac{2\pi i}{\cancel{L_x}}\cancel{L_x}\cdot\sum_y n_{a,y}\right) T_x|\psi_0'\rangle$$

$$T_x|\psi_0'\rangle = e^{ip_0}|\psi_0\rangle$$

$$\therefore\ \Delta p = \frac{2\pi i}{L_x}\sum_{\vec r} n_{\vec r}$$

$$= \frac{2\pi i}{\cancel{L_x}}\, \nu \times \cancel{L_x} \times L_y$$

$$= 2\pi i\, L_y \times \frac{p}{q}$$




$$a\, n_{2a,y} + (2a)\, n_{3a,y} + \dots (L-1)a\, \underline{\underline{n_{La,y}}} + (La)\, \underline{\underline{n_{a,y}}}$$

if $\gcd(q, L_y) = 1$, then

$$\Delta p \neq 2\pi$$

so $\exists$ a degen. orthogonal state wrt $|\psi_0\rangle$.

# flux threading can be done "q" times, leading to <u>q low lying states.</u>

# GS. deg., thus, is an interaction-strength independent robust property of a Q. system.

# <u>Assumptions</u>:- Translational × U(1)

## Relation to $s=\frac{1}{2}$

spin $\frac{1}{2}$ to hard core bosons

$$S^+ = b^+ \qquad S^- = b \qquad S^z = b^+ b - \frac{1}{2}$$

$\because -\frac{1}{2} < S^z < \frac{1}{2} \Rightarrow \quad 0 < b^+ b < 1$

when $\sum S^z_x = 0$ ( say AFM in $d=1$ ) $\Rightarrow \boxed{\sum_{x=1}^{L} b^+(x)\, b(x) = \frac{L}{2}}$
or
singlet state

$\Rightarrow$ Bosons at $\frac{1}{2}$ filling

Oshikawa's $q=2$ case

$\therefore \exists$ one more state (in agreement with LSM)

# A fermionic $S=\frac{1}{2}$ representation:-

$$\vec{S} = f^+ \frac{\vec{\sigma}}{2} f \qquad f \rightarrow \downarrow \uparrow$$ (Called Schwinger-Wigner representation)

$$S^a = f_\alpha^+ \frac{\sigma^a_{\alpha\beta}}{2} f_\beta$$ generally $f$ can be either boson/fermion.

$$[S^a, S^b] = \left[f_\alpha^+ \frac{\sigma^a_{\alpha\beta}}{2} f_\beta,\ f_\gamma^+ \frac{\sigma^b_{\gamma\delta}}{2} f_\delta\right]$$

$$= \frac{1}{4} \sigma^a_{\alpha\beta} \sigma^b_{\gamma\delta} [f_\alpha^+ f_\beta, f_\gamma^+ f_\delta]$$

$$= \frac{1}{4} \sigma^a_{\alpha\beta} \sigma^b_{\gamma\delta} (f_\alpha^+ f_\beta f_\gamma^+ f_\delta - f_\gamma^+ f_\delta f_\alpha^+ f_\beta)$$

$$- \cancel{\alpha^+ \gamma^+ \beta\delta} + \delta_{\beta\gamma}\, \alpha^+ \delta + \cancel{\gamma^+ \alpha^+}\, \delta\beta$$
$$- \delta_{\delta\alpha}\, \gamma^+ \beta$$

$$= \frac{1}{4} \sigma^a_{\alpha\beta} \sigma^b_{\gamma\delta} (\delta_{\beta\gamma} f_\alpha^+ f_\delta - \delta_{\delta\alpha} f_\gamma^+ f_\beta)$$

$$= \frac{1}{4} \left(\sigma^a_{\alpha\beta} \sigma^b_{\beta\delta} f_\alpha^+ f_\delta - \sigma^a_{\alpha\beta} \sigma^b_{\gamma\alpha} f_\gamma^+ f_\beta\right)$$

$$\sigma^a_{\alpha\beta} \sigma^b_{\beta\gamma} f_\alpha^+ f_\gamma - \sigma^a_{\alpha\beta} \sigma^b_{\gamma\alpha} f_\gamma^+ f_\beta$$

$$= \frac{1}{4}\left[ i\epsilon^{abc} \sigma^c_{\alpha\delta} f_\alpha^+ f_\delta - (i\epsilon^{abc}\sigma^c_{\beta\gamma})^* f_\gamma^+ f_\beta \right]$$

$$= \frac{1}{4}\left[ i\epsilon^{abc} f_\alpha^+ \sigma^c_{\alpha\delta} f_\delta + i\epsilon^{abc}\sigma^c_{\gamma\beta} f_\gamma^+ f_\beta \right]$$

$$= i\epsilon^{abc} \quad f_\alpha^+ \frac{\sigma^c_{\alpha\delta}}{2} f_\delta = i\epsilon^{abc} S^C$$

<u>Note</u>:- A recent work by Vishwanath et al describes how to relate Kramers theorem & LSM.

Essentially, one can exploit Kramers theorem to comment on ground state degeneracy of translationally invariant <u>systems</u>.

# Some spin representations

## ① Holstein primakoff bosons

main idea: In a broken symmetry phase, $\langle S^i \rangle$ is non-zero for atleast some $i \in x, y, z$.

→ natural to describe ordered phase in terms of small spin fluctuations about their expectation values.

HP bosons:

$$S^+ = \sqrt{2S - n_b}\, b \qquad \because \quad \underset{b^+b = 0}{\uparrow\uparrow\uparrow\uparrow} \xrightarrow{S^-} \underset{b^+b \neq 0}{\uparrow\uparrow\nearrow\uparrow}$$

$$S^- = b^+ \sqrt{2S - n_b}$$

$$S^z = S - b^+ b$$

subspace:- $\{|n_b\rangle\} = \{ |0\rangle, \ldots |2S\rangle \}$

$n_b > 2S \rightarrow$ spurious states

spin operators satisfy $[S^a, S^b] = i\epsilon^{abc} S^c$

$\Downarrow$ & $S^2 = S(S+1)$

don't connect physical to unphysical subspaces.

$$\sqrt{2S - n_b} = \sqrt{2S} - \frac{n_b}{2\sqrt{2S}} + \cdots \rightarrow \text{"semiclassical expansion"}$$

↳ expansion around $\langle S^z \rangle$

$\Downarrow$

linearize $\mathcal{H}$ upto $b^+b$, "linear spinwave theory"

## ② Schwinger Bosons :- • Useful for symmetric phase of $\mathcal{H}_{\text{heisenberg}}$

(or non-ordered phase, like spin liquids)

$S^+ = a^\dagger b \qquad S^- = b^\dagger a \qquad S^z = \dfrac{n_a - n_b}{2}$

$a, b \rightarrow$ indpt

$\downarrow$

correlated by constraint

$\boxed{\text{s.t. } n_a + n_b = 2S}$

generic spin states are given by

$$|S, m\rangle = \frac{(a^\dagger)^{S+m} (b^\dagger)^{S-m}}{\sqrt{(S+m)!}\sqrt{(S-m)!}} |0\rangle$$

$\hookrightarrow$ boson vacuum.

for $S = 1/2$,

$|\uparrow\rangle = a^\dagger |0\rangle$

$|\downarrow\rangle = b^\dagger |0\rangle$

$S^+ = a^\dagger b \qquad S^- = b^\dagger a$

③ Schwinger - Wigner mapping

denote $\vec{S} = f^\dagger \dfrac{\vec{\sigma}}{2} f \qquad f = (f_\uparrow \; f_\downarrow) \rightarrow$ can be bosons/fermions.

(same as schwinger bosons, except fermions are also allowed)

④

# Large S approach to Quantum magnetism

# useful to study ordered phases of matter. One maps s to bosons,
& phases of bosons corresp. to diff ordered states.

# Approx gets better at large S.

## Holstein Primakoff representation

$$S^+ = \left(\sqrt{2S - b^+ b}\right) b$$

$0 \leq b^+ b \leq 2S$

$$S^z = S - b^+ b$$

# Practice problems

3. [2+2+2+4+5+5+5=25 marks] Consider the following model for a spin $S = 1/2$ particle,

$$H = -J \sum_j S_j^z S_{j+1}^z - h \sum_j S_j^x, \tag{1}$$

where $S_j^{z,x}$ are the $z, x$ projections of the spin operator at site $j$, $J$ and $h$ are positive constants, and the total number of sites is $L$. Denoting the two possible eigenstates of the $S_j^z$ as $|\uparrow_j\rangle$ and $|\downarrow_j\rangle$, answer the following.

(a) Write down the possible ground state(s) when $h = 0$.

(b) Write down the possible ground state(s) when $h \to \infty$, i.e., when $J$ can be neglected.

(c) Interpret physically the above two cases in terms of spin polarization. Support your conclusion by calculating the total magnetization in the $z$-direction, $\langle \sum_j S_j^z \rangle$, where $\langle \cdots \rangle$ denotes the average in the ground state at $T = 0$. If you have done all the steps so far correctly, you will realize that the two limits describe two different phases.

(d) Let us calculate the excitation spectrum of this model. In class we found that excitations of the Heisenberg model can be studied by using the Holstein-Primakoff transformation where the excitations are treated as Bosons. The spin-1/2 case considered here is a rather special case where one can treat it as a Fermionic



object. Why? Because you can flip a spin-1/2 object only once unlike higher spins. Thus, we can imagine that the down spin is the Fermionic vacuum, and the up spin is a state where one Fermion has been created:

$$|\downarrow\rangle \equiv |0\rangle, \quad |\uparrow\rangle \equiv c^\dagger|0\rangle, \quad \text{implying} \quad \underline{S^+ \equiv c^\dagger}, \quad \underline{S^- \equiv c.} \tag{2}$$

Using this, show that

$$S^z \equiv c^\dagger c - \frac{1}{2} \quad \text{and} \quad S^x \equiv \frac{1}{2}\left(c^\dagger + c\right). \tag{3}$$

(e) Now we can use the above expressions in $H$ and calculate. However, before doing so, we need to go through two more steps. First, because the total energy does not depend on how we define our coordinate axes, we can interchange $x \leftrightarrow z$ in Eq. (1) to write

$$H = -J \sum_j S_j^x S_{j+1}^x - h \sum_j S_j^z. \tag{4}$$

Second, we need to know how to properly extend Eqs.(3) to the lattice. This is achieved by the following set of transformations:

$$S_j^z = c_j^\dagger c_j - \frac{1}{2}$$

$$S_j^+ = c_j^\dagger \exp\left(i\pi \sum_{i=1}^{j-1} c_i^\dagger c_i\right)$$



Page 2 of 2

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$$S_j^- = \exp\left(-i\pi \sum_{i=1}^{j-1} c_i^\dagger c_i\right) c_j, \tag{5}$$

Use Eqs. (5) in (4) to show that, in terms of the Fermionic operators $c_j$ and $c_j^\dagger$, $H$ can be expressed as

$$H = \frac{J}{4} \sum_j (c_j - c_j^\dagger)(c_{j+1} + c_{j+1}^\dagger) - h \sum_j c_j^\dagger c_j. \tag{6}$$

(f) As usual, go to the Fourier space and rewrite $H$. If you have done correctly, you will find it of the form

$$H = \sum_{\mathbf{k}} \left[\gamma_{1\mathbf{k}} \left(c_{\mathbf{k}}^\dagger c_{\mathbf{k}} - c_{-\mathbf{k}} c_{-\mathbf{k}}^\dagger\right) - i\gamma_{2\mathbf{k}} \left(c_{\mathbf{k}}^\dagger c_{-\mathbf{k}}^\dagger - c_{-\mathbf{k}} c_{\mathbf{k}}\right)\right]. \tag{7}$$

Find $\gamma_{1\mathbf{k}}$ and $\gamma_{2\mathbf{k}}$.

(g) Finally, diagonalize the above expression to find the excitation spectrum. Comment and interpret whether the spectrum is gapped or not.

Re.

$$S_j^x = \frac{S^+ + S^-}{2}$$

$$= \left(\frac{1}{2}\right)\left[c_j^+ \exp\left(i\pi \sum_{i=1}^{j-1} n_i\right) + c_j \exp\left(-i\pi \sum_{i=1}^{j-1} n_i\right)\right]$$

now

$$\exp(\pm i\pi n_i) = 1 + \pm i\pi n_i + \frac{(\pm i\pi)^2}{2!} n_i^2 + \cdots$$

$$= 1 + \left(\pm i\pi + \frac{(\pm i\pi)^2}{2!} - \cdots\right) n_i \quad \left[\because n_i^2 = n_i\right]$$

$$= 1 + \left(e^{\pm i\pi} - 1\right) n_i$$

$$= 1 - 2n_i$$

$$\therefore S_j^x = \frac{1}{2}\left[c_j^+ + c_j\right] \prod_{i=1}^{j-1} (1 - 2n_i)$$

$$\therefore S_j^x S_{j+1}^x = \frac{1}{4}(c_j^+ + c_j)(c_{j+1}^+ + c_{j+1}) \prod_{i=1}^{j-1}(1-2n_i) \times \prod_{i=1}^{j}(1-2n_i)$$

now

$$(1-2n_i)^2 = 1 - 4n_i + 4n_i = 1$$

$$\Rightarrow S_j^x S_{j+1}^x = \frac{(c_j^+ + c_j)(c_{j+1}^+ + c_{j+1})}{4}\left\{\prod_{i=1}^{j-1}(1-2n_i)^2\right\} \times (1-2n_j)$$

$$= \frac{(c_j^+ + c_j)(c_{j+1}^+ + c_{j+1})}{4}(1-2n_j)$$

now let $m_j = c_j + c_j^+$

$d_j = c_j - c_j^+ \qquad m_j^2 = 1 \;\; \& \;\; m_j d_j = 2n_j - 1$

$$\Rightarrow \quad S_j^x S_{j+1}^x = \frac{1}{4} \quad m_j \, m_{j+1} \times -1 \times m_j d_j$$

$$= \left(\frac{-1}{4}\right)(-1) \, m_j^2 \, m_{j+1} \, d_j$$

$$= -\frac{1}{4} d_j m_{j+1} = -\frac{1}{4} \left(c_j - c_j^+\right)\left(c_{j+1} + c_{j+1}^+\right)$$

$$\therefore \; \mathcal{H} = -J \times -\frac{1}{4} \sum_j \left(c_j - c_j^+\right)\left(c_{j+1} + c_{j+1}^+\right) - h \sum c_j^+ c_j$$

<u>**$\mathcal{F}$f:-**</u> $\quad c_j = \dfrac{\sum_k e^{ikj} c_k}{\sqrt{L}}$

$$\mathcal{H} = \frac{+J}{4L} \sum_k \sum_j \left(c_{k_1} e^{ik_1 j} - c_{k_2}^+ e^{-ik_2 j}\right)\left(c_{k_3} e^{ik_3 j} e^{ik_3} + c_{k_4}^+ e^{-ik_4 j} e^{-ik_4}\right)$$

$$- h \sum c_k^+ c_k$$

$$= \left(\frac{J}{4}\right)\left\{ \sum_k c_k \left( c_{-k} e^{-ik} + c_k^+ e^{-ik}\right) - \sum_k c_k^+ \left( c_k e^{ik} + c_{-k}^+ e^{ik}\right) \right\}$$

$$- h \sum c_k^+ c_k$$

$$= \frac{J}{4} \sum_k c_{-k} \left( c_k e^{ik} + c_{-k}^+ e^{ik}\right) - \sum_k c_k^+ \left( c_k e^{ik} + c_{-k}^+ e^{ik}\right)$$

$$- h \sum c_k^+ c_k$$

$$= \sum_k -\frac{J}{4} e^{ik}\left(c_k^+ c_k - c_{-k} c_{-k}^+\right) - \frac{J}{4} e^{ik}\left(c_k^+ c_{-k}^+ - c_{-k} c_k\right)$$

$$-h \sum c_k^\dagger c_k$$

$$= \sum_k \left(-\frac{J}{2}\cos k - h\right)\left(c_k^\dagger c_k - c_{-k} c_{-k}^\dagger\right)$$

$$-\frac{J}{4} i \sin k \left(c_k^\dagger c_{-k}^\dagger - c_k c_{-k}\right)$$

$$2\sum_k c_k^\dagger c_{-k}^\dagger e^{ik}$$

$$= \sum_k c_k^\dagger c_{-k}^\dagger e^{ik} - \sum_k c_{-k}^\dagger c_k^\dagger e^{ik}$$

$$= \sum c_k^\dagger c_{-k}^\dagger (2 i \sin k)$$

$$\therefore \gamma_1(k) = \left(-\frac{J\cos k}{2} - h\right)$$

$$\gamma_2 = \frac{J}{4} \sin k$$

<u>Eq.</u> Looks like $H_{BCS}$. In "nambu" basis

$$H = \sum_k \left(c_k^\dagger \quad c_{-k}\right) \underbrace{\begin{bmatrix} \gamma_k^2 & -i\gamma_k^2 \\ i\gamma_k^2 & -\gamma_k^1 \end{bmatrix}}_{\gamma^1 \sigma_z + \gamma^2 \sigma_y} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2^\dagger \end{pmatrix}$$

$$\Vert$$

$$E_k = \pm \sqrt{\gamma_k'^2 + \gamma_k^{2\,2}}$$

$$= \sqrt{\left(\frac{J}{2}\cos k + h\right)^2 + \left(\frac{J}{4}\sin k\right)^2}$$

which is gapped if $h \neq \pm \frac{J}{2}$

if $h = \frac{J}{2}$, then @ $k = \pi$

$$E_k \approx \left[\frac{J}{4} k^2 + 0^2\right]^{1/2} \approx \left(\sqrt{\frac{J}{4}}\right)|k|$$

$$\mathcal{H} = \sum E_k \, \gamma_k^{1\dagger} \gamma_k^1 - \sum E_k \, \gamma_k^2 \, \gamma_k^{2\dagger}$$

$$= \sum_k E_k \left( \gamma_k^{1\dagger} \gamma_k^1 + \gamma_k^{2\dagger} \gamma_k^2 \right) - \sum E_k$$