High-Low temperature expansions of the Ising Model and Duality

Course project for Advanced Statistical Mechanics (PH 543)
Guru Kalyan Jayasingh

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The main idea of this exposition is highlight the study of lattice models (for e.g. ising model) by using expansions around exactly solvable limits of the same. This helps us arrive at dualities between different lattice models, ultimately culminating in a non-local order parameter harbouring phase transition. I've followed Kardar's reference on stat field theory in making these.

1 Low Temperature Expansion

We work with a d-dimensional ferromagnetic ising model ($d > 2$)

$$-\beta\mathcal{H} \;=\; K\sum_{<ij>}\sigma_i\sigma_j, \quad K = \beta J > 0 \tag{1}$$

Now look at the low temperature limit. In this case all the spins have aligned (WLOG we can take $\sigma_i = +1$) and excitations around the same correspond to overturning an up spin to down. Now the lowest energy excitation would be turning one spin down, costing an energy of 2K*2d with N ways of doing it. The next higher excitation would be to turn 2 spins. To keep the cost minimum, we turn a dimer instead of two well-separated disjoint spins. We can use this idea to keep iterating the procedure and compute the partition function as

$$Z \;=\; 2e^{dNK}\big[1 + Ne^{-4dK} + dNe^{-4(2d-1)K} + \frac{N(N-2d-1)}{2}e^{-8dK} + ...\big] \tag{2}$$

The factor of 2 is due to two-fold degeneracy while the 4th order term captures the disjoint spin flips. As this is at low temperature, $\beta \to \infty$, hence we need to keep only terms of the kind e^{-cK}. This can be done by looking at the free energy per site

$$-\beta f \;=\; \frac{lnZ}{N} \;= dk + \frac{1}{N}ln\big[1 + Ne^{-4dK} + dNe^{-4(2d-1)K} + \frac{N(N-2d-1)}{2}e^{-8dK} + ...\big] \tag{3}$$

$$= dK + e^{-4dK} + de^{-4(2d-1)K} - \frac{2d+1}{2}e^{-8dK} + ... \tag{4}$$

Note that there's a cancellation of terms of order N^2. This can be explicitly checked, however a quicker way to see is that f being intensive, this cancellation was guaranteed. This then settles the cancellation to be true for higher orders as well.

2 High temperature Expansion

While low temp expansion doesn't work for continuous systems (for e.g. XY model) due to presence of Goldstone modes, high temperature expansion is compatible with both. To effect this expansion, the high T state is rightly chosen to be that of independent spins. Expansion in β is done around this state giving us

$$Z \;=\; tr(e^{-\beta\mathcal{H}}) \;=\; tr\left[1 - \beta\mathcal{H} + \frac{\beta^2\mathcal{H}^2}{2} + ...\right] \tag{5}$$

$$-\beta f \;=\; \frac{lnZ}{N} \;=\; \frac{lnZ_0}{N} - \beta\frac{\langle H\rangle_0}{N} + \frac{\beta^2}{2}\frac{\langle H^2\rangle_0 - \langle H\rangle_0^2}{N} + ... \tag{6}$$

where $\langle\rangle_0$ is the average taken over independent spins. For the ising model, however, there's a more powerful graphical approach yielded if we take $t = tanhK$ (which $\to 0$ at high T) as our expansion parameter. Since $(\sigma_i\sigma_j)^2 = 1$, this yields

$$e^{K\sigma_i\sigma_j} = coshK + sinhK \cdot \sigma_i\sigma_j = cosh\ K\ (1 + t\sigma_i\sigma_j) \tag{7}$$

Now the partition function becomes

$$Z = \sum_{\{\sigma_i\}} e^{K \sum_{\langle i,j\rangle} \sigma_i\sigma_j} = (coshK)^{\#ofbonds} \sum_{\{\sigma_i\}} \prod_{\langle i,j\rangle} (1 + t\sigma_i\sigma_j) \tag{8}$$

For x no. of lattice bonds, the total no. of pairs generated are 2^x. Now comes the crucial observation: Since t is small, we can essentially expand the above product in a polynomial of t. What would a given power of t mean then? The answer is easy if we think as follows: every power of t comes with a $\sigma_m\sigma_n$ factor - Denote this graphically by joining a line from m to n site and putting a factor of t for that. This is shown in Fig.1.

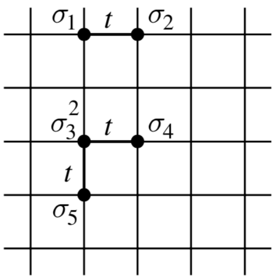

Figure 1: Labelling lattice bonds

For a term containing t^m, we can see that it'll have m lattice bonds in it. The $\sum_{\{\sigma_i\}}$ will only give non-zero values for terms that have σ_l^2, hence we require that for each such t^m term, the corresponding σ_j appearing must have even powers. This would mean that every lattice site contributing to the t^m term should be evenly linked. This restricts the terms greatly, namely:

- m = even (employ distance argument to convince)
- Graphs have to be closed i.e. can't have open ends (since the sum vanishes for that end)

Hence only surviving graphs have even no. of lines passing through each site. Hence the partition function could be rewritten as

$$Z = 2^N \cdot (cosh(K))^{N_b} \sum_{closed\ graphs} t^{number\ of\ lattice\ bonds\ in\ each\ graph} \tag{9}$$

Note: Having no lattice bond also constitutes a graph i.e. $\exists$ a 1 contibuting to the sum above. Also the 2^N in the sum results from summing over all sites, whether or not they appear as σ. The closed graphs can be easily enumerated by drawing them on a lattice as shown in Fig.2.

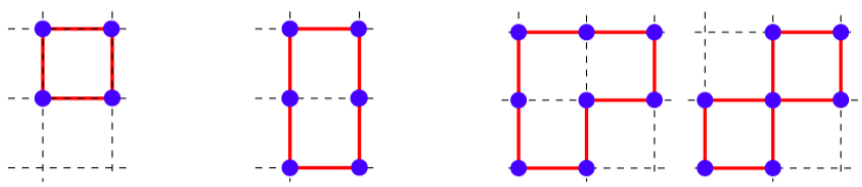

Figure 2: Closed graphs enumeration

In d-dimensions, we can write

$$Z = 2^N (coshK)^N \left[1 + \frac{d(d-1)N}{2} t^4 + d(d-1)(2d-3)t^6 + ...\right] \tag{10}$$

where the 2nd term corresponds to choosing a square in d-dimensions, no. of ways of which are $\frac{d(d-1)}{2}$.

3 Self-Duality of 2D ising model

The above expansions can lead to a duality between high-low temp expansions of the 2D ising model, called the Kramers-Wannier's duality. This can be easily sen by comparing the partition function expansions:

1. Low temperature:

$$Z \ = \ e^{2NK}\left[1+Ne^{-4*2K}+2Ne^{-6*2k}+....\right] \tag{11}$$

$$= e^{2NK} \sum_{Islands\ of\ (-)\ droplets} e^{-2K\cdot\ perimeter\ of\ island} \tag{12}$$

2. High Temperature:

$$Z \ = 2^{N}(coshK)^{2N} \sum_{graph\ with\ 2\ or\ 4\ lines\ per\ site} (tanhK)^{length\ of\ graph} \tag{13}$$

The crucial observation is this: in 2D square lattice, the boundary of islands of ↓ spin droplets can only have sites with 2/4 links emanating from them - hence they make an acceptable closed graph.This ensures that there's a duality in the expansions given by the above, with duality relation

$$e^{-2\tilde{K}} \ = tanh\ K \tag{14}$$

With the hindsight of a phase transition, we can compute the critical coupling (i.e. $\beta_c J$): From the low T- high T duality, we know that $\mathcal{F}$ has a singular part which can be expressed as a function $g(x)$, where x is $e^{-2\tilde{K}}$ for low T and $tanh\ K$ for high T. Therefore, a singularity in $\mathcal{F}$ at some $\tilde{K}$ is reflected in the same being at some high temp K. But we know that there's only one singularity: at critical point. Hence the critical coupling must map to itself , which leads to

$$e^{-2K_c} \ = tanh\ K_c \tag{15}$$

which gives $K_c \ = -\frac{1}{2}ln(\sqrt{2}-1)$. Note the strength of this result over MFT approach.

4 3D ising model and Lattice gauge theory

The low temperature expansion leads to

$$Z \ = e^{-3NK}\left[1+Ne^{-2K\times 6}+3Ne^{-2K\times 10}+....\right] \tag{16}$$

$$Z = e^{3NK} \sum_{islands\ of\ (-)\ spins} e^{-2K\times area\ of\ island's\ boundary} \tag{17}$$

while the high temperature expansion yields

$$Z = 2^{N} coshK^{3N} \sum_{with\ 2,4,6\ lines\ per\ site} (tanh\ K)^{number\ of\ lines} \tag{18}$$

Diagrammatically, it looks like fig.(3). We see that the high temperature has two problems: It incorporates all the closed graphs from 2d ising (which don't enclose any − spin islands in 3D) and that it allows for these "square-like" patches to join up and produce open chain (see the last fig in fig.(3, high T series). Hence there's no self-duality for 3D ising model. Can it be dual to another model?

As it happens, this is can be done. The corresponding dual (for a reasoning on how to arrive at it intutively, see kardar's relevant sections) is given by

$$-\beta\mathcal{H}_{3D\ dual} \ = \ K \sum_{plaquettes} \tilde{\sigma}^{i}_{P}\tilde{\sigma}^{j}_{P}\tilde{\sigma}^{k}_{P}\tilde{\sigma}^{l}_{P} \tag{19}$$

where a plaquette is a square on the SC lattice. This is an example of a Z_2 lattice gauge theory. The name arrives because, in addition to Z_2 symmetry of the lattice, we have a local/gauge symmetry as follows: Pick a site and revert all signs of all ising spins that emanate from it (6 in this case).

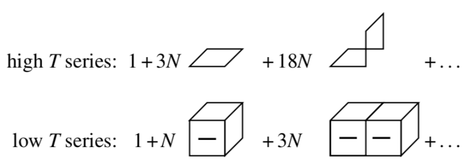

Figure 3: 3D ising expansions

Now since this changes the sign of 2 spins in each adjoint
plaquette, $\mathcal{H}$ remains invariant.
Now **Elitzur's theorem** states that spontaneous breaking of local symmetries isn't possible. What the means is that proceeding similar to the ising case, if we define a local order parameter (say m, magnetization for this case),it cannot characterise spontaneous symmetry breaking type phase transition. What can be shown is that (see Muramatsu's notes) as we switch on the symmetry breaking field h, we see that $\langle\tilde{\sigma}(m,\hat{e}_i)\rangle$ (for m^{th} site and i^{th} direction link) will be a continuous function of h and vanishes as $h\to 0$. The physical reason why this happens is that due to local transformations, the system only has finite change in energy in presence of h, hence no singularities arise as we take $h\to 0$.

5 Phase transitions via a Non-Local order parameter

While the usual 3D ising model shows a phase transition (see Peierl's arguments), for it's dual to show the same, it can't proceed via a local order parameter. Hence, Wegner suggested that the same happens via a non-local order parameter. For this case, a differing asymptotic behaviour of correlation function differentiates the two phases from each other.
Since only gauge invariant quantities will have non-zero values, one consrtucts a correlation function called *Wilson loop* defined as

$$C_s \ = \langle Product\ of\ \tilde{\sigma}\ around\ the\ loop\rangle = \langle\prod_{i\varepsilon S}\tilde{\sigma}_i\rangle \tag{20}$$

One can check that this is gauge invariant, since gauge transformation changes two bonds in S, product of which remains invariant. We can now look at the behaviour of wilson loop at high and low temperatures. For high temperatures, partition function shall is a sum of all graphs constructed from plaquettes with S as their boundary (they're really sheets on lattice, with a lattice area of A_s and perimeter P_s). Each plaquette contributes a $tanh\ K$, so the contribution can be written

$$\begin{aligned} C_s \ &= \frac{1}{Z}\sum_{\{\tilde{\sigma}\}}\prod_{i\varepsilon S}\tilde{\sigma}_i e^{K\sum\tilde{\sigma}_P^i\tilde{\sigma}_P^j\tilde{\sigma}_P^k\tilde{\sigma}_P^l} &(21)\\ &= (tanhK)^{A_s}\big[1+\mathcal{O}\big(\tanh\ K^2\big)+...\big] &(22)\\ &= exp(-f(tanh\ K)\times A_s) &(23)\end{aligned}$$

Note that A_s is the minimal area enclosed by the graph while the squared term comes from taking the sheet with minimal area and adding just one more adjoining plaquette, thus making 3 plaquettes, but 1 gets absorbed in the prefactor before the sum, hence we get squared terms.
The low temperature expansions starts (not with an ordered state) with lowest energy configuration.However, $\exists\ N_G = \ 2^N$ such states, related to each other by gauge transformation. Since C_s is gauge independent, it is sufficient to look at one of the ground states, namely with all $\tilde{\sigma}_i = +1$ (Caution: this is A lowest energy state, isn't necessarily the true thermal state). The minimum energy excitations correspond to flipping a link/bond. This will cost 8K worth of energy with 3N ways of effecting it. Also, P_s, defined previously, denotes the of bonds present on the perimter of S. We can then write

$$C_s = \frac{N_G}{N_G}\frac{e^{3n\tilde{K}P}\big[1+(3N-P_s)e^{-2\tilde{K}\cdot 4}+(-1)P_s e^{-2\tilde{K}\times 4}......\big]}{e^{3n\tilde{K}P}\big[1+3Ne^{-2\tilde{K}\cdot 4}+......\big]} \tag{24}$$

where N_G comes from the gauge invariance induced degeneracy. The first term in the numerator corresponds to not flipping any link on S, while the 2nd term denotes P_s ways of flipping a link on S. This can be written more succintly as

$$\begin{aligned} C_s = \ &1-2P_s e^{-8\tilde{K}} &(25)\\ = \ &exp\big[-2e^{-8\tilde{K}}P_s\big] &(26)\end{aligned}$$

Thus the asymptotic dependence of C_s is different at high and low temperatures: at high T, it's decay is controlled by Area while at low T it's decay is controlled by Perimeter. The phase transition marks the change from one type to another, and by duality, has same singularities as Ising model.