High and low temperature expansions of the ising model and Duality

Topics

- 2d ising model expansions
- Duality in 2d ising model
- Duality in 3D ising
- Example

2D ising model

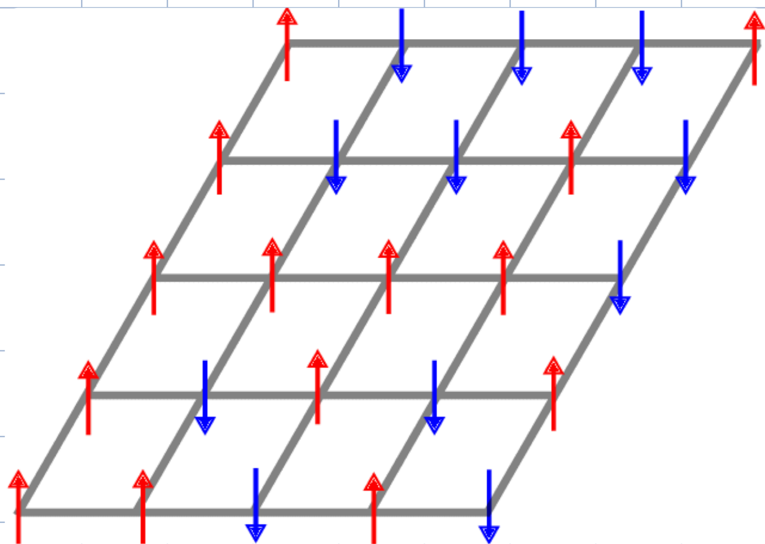


Figure: Ising Model

$$-\beta \mathcal{H} = K \sum_{\langle ij \rangle} \sigma_i \sigma_j \quad , K = \beta J > 0$$

① 2D Ising model shows phase transition

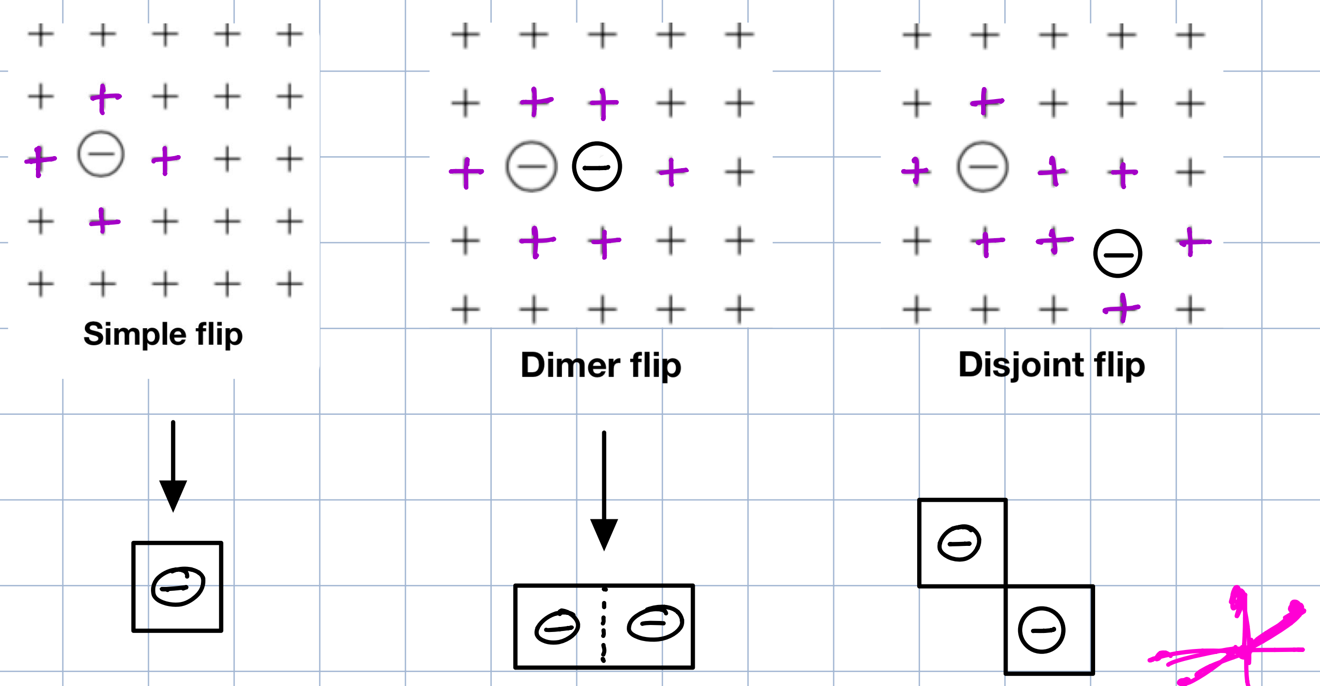
② Has a global discrete spin symetry

③ 2D/ Higher D ising still show phase transition.

Ising model low T exp.

- Take the original state to be ordered $S_i = +1$ and then look at small excitations(spin flips) around it.

Excitations:



$\ominus\ominus \longrightarrow \frac{N \times 2d}{2} = Nd$

$$Z = 2 e^{2NK}\left[1 + N e^{-2K\times 4} + 2N e^{-2K\times 6} + \cdots\right]$$

$$= e^{2NK} \sum_{\substack{\text{islands} \\ \text{of -ve} \\ \text{spin}}} \exp\left[-2K \cdot \text{perimeter of island}\right]$$

High T expansion

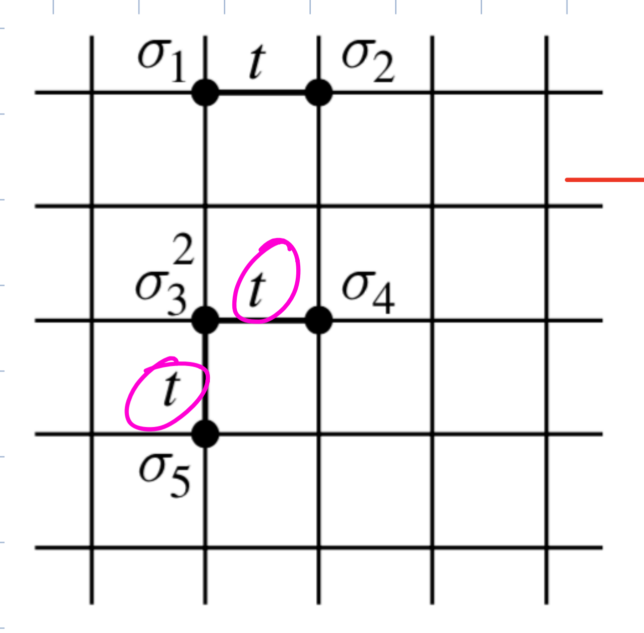
- For the ising model, a natural way to analyse high temperature expansion is to look at the parameter tanh K.

$$\exp(K\sigma_i\sigma_j) = \cosh K + (\sinh K)\,\sigma_i\sigma_j = \cosh K\left[1 + \underbrace{t}_{\tanh K}\sigma_i\sigma_j\right]$$

$$\therefore\ Z = \sum_{\{\sigma_i\}} \exp\Big\{K\sum_{\langle i,j\rangle}\sigma_i\sigma_j\Big\} \qquad (K = \beta J > 0)$$

$$= (\cosh K)^{\#\text{ of bonds}} \sum_{\{\sigma_i\}}\prod_{\langle i,j\rangle}\underbrace{(1+t\sigma_i\sigma_j)}_{a_0 + a_1 t + a_2 t^2}$$

$\sigma_i\sigma_j$

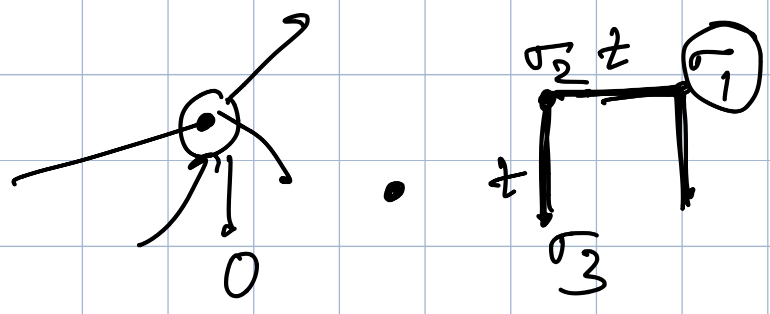


$\sigma_i\sigma_j\sigma_n$ - - -

denote every $t\sigma_i\sigma_j$ term by a line.

• Every t^m term will have m bonds in it.

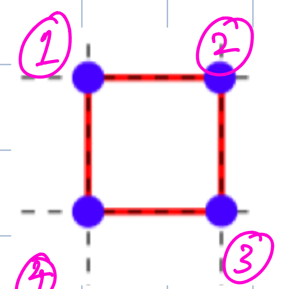
e.g.:- $t\,\underline{\sigma_1\sigma_2}\;\;t\,\underline{\sigma_2\sigma_3} = t^2\,\sigma_1(\sigma_2)^2\sigma_3$



$$= t^2\,\sigma_1\sigma_3 \;\xrightarrow{\sum_{\sigma_1,\sigma_3}}\; 0$$

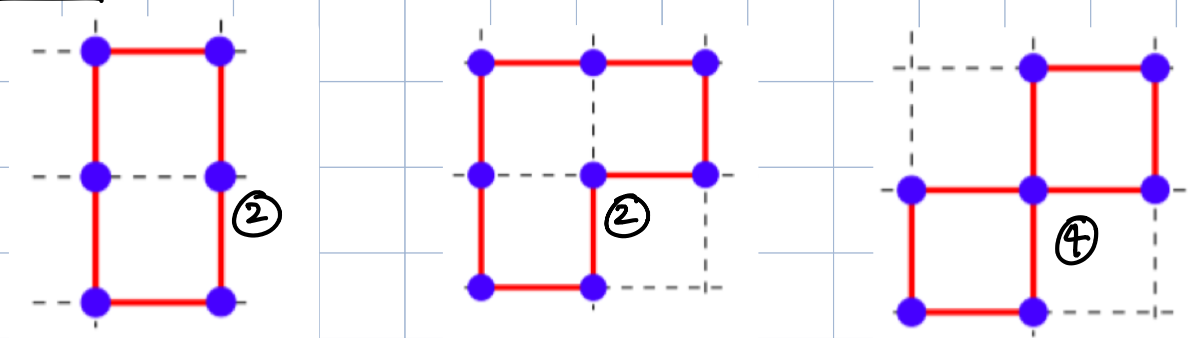
∴ Sum will only be non-zero if power of every σ_i is even i.e. even linking of lattice site. (2 or 4 links)

e.g.:- $t\sigma_1\sigma_2\;\; t\sigma_2\sigma_3\;\; t\sigma_3\sigma_4\;\; t\sigma_4\sigma_1$



• So only closed graphs contribute

<u>Some more examples</u>



Duality in 2d ising

- Look at the series expansions.

For the low temperature case we have,

$$Z_{low} = 2\, e^{2NK}\left[1 + N\, e^{-2K\times 4} + 2N\, e^{-2K\times 6} + \cdots\right]$$

$$= e^{2NK} \sum_{\substack{\text{islands} \\ \text{of -ve} \\ \text{spin}}} \exp\left[-2K \cdot \text{perimeter of island}\right]$$

For the high temperature case, we subsequently have

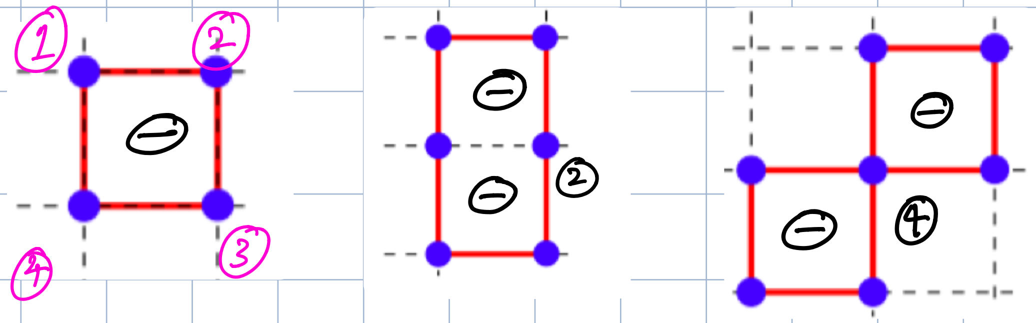
$$Z_{high} = 2^N (\cosh K)^{2N}\left[1 + N(\tanh K)^4 + 2N(\tanh K)^6 + \cdots\right]$$

$$= 2^N (\cosh K)^{2N} \sum_{\substack{\text{all 2 or} \\ \text{4 linked graphs}}} (\tanh K)^{\text{length of graph}}$$

<u>Duality</u>

$$e^{-2\tilde{K}} \equiv \tanh K \Rightarrow \tilde{K} = D(K) \equiv -\tfrac{1}{2} \tanh K$$

Why duality?



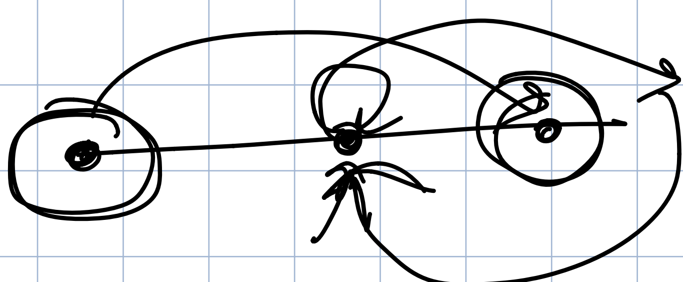
- Hence it maps a low temp to a corresponding high temp (and vice versa)

$$-\beta f = \frac{\ln Z}{N} = 2K + g(e^{-2K}) = \text{const.} + g(\tanh K)$$

- Therefore if g is singular at k1, it should also be singular at a corresponding k2. Since the model is analytic everywhere except at critical point, it must be self dual at the critical coupling

$$e^{-2K_c} = \tanh K_c$$

$$\Rightarrow \underline{K_c = 0.441} \qquad (K = \beta J)$$



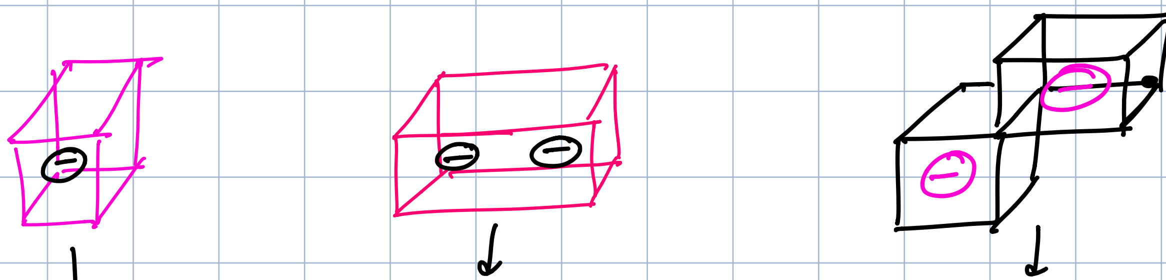
$$K_c = \underline{\beta_c J}$$

3D ising model duality

- Low temperature expansion

$$Z = e^{3NK}\left[1 + N e^{-2K\times 6} + 3N e^{-2K\times 10} + \cdots\right]$$

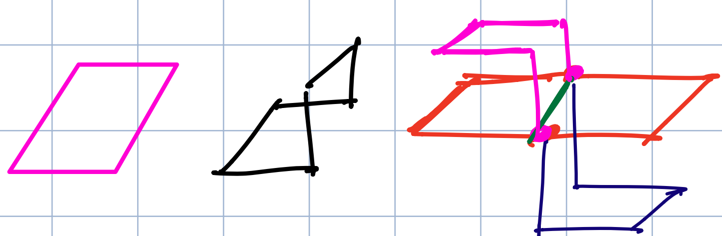
$$= e^{3NK} \sum_{\substack{\ominus \text{ spin} \\ \text{islands}}} \exp\left[-2K \times \text{Area surrounding the -ve spin}\right]$$



$e^{-2K\times 6}$ $\quad$ $e^{-2K\times 10}$ $\quad$ $e^{-2K\times 12}$

- High temperature expansion

$$Z = 2^N (\cosh K)^{3N} \sum_{\substack{\text{sites} \\ \text{with } 2,4 \text{ or } 6 \\ \text{links}}} (t)^{\text{perimeter of the surface}}$$



they don't enclose any -ve spin islands

No duality. Might be dual to some other model.

What features should this dual model have?

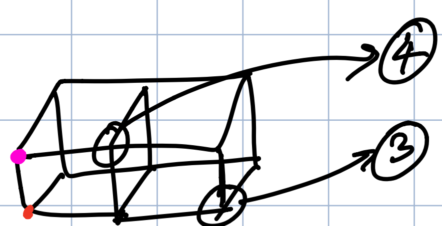
1. Use areas as building blocks instead of bonds.

Why?

∵ Low Temp $= e^{-2K \times \text{area}}$

6 10

2. Plaquettes must only contribute when they enclose a closed surface. What will ensure this?



$4 \xrightarrow{\Sigma} 1$

$3 \xrightarrow{\Sigma} 0$

So spins at lattice site won't allow this.

∴ Place ising spins on bonds.

(why?) → (2,4) idea in 2D ising

$\therefore$ By analogy with Ising model we write

$$\tilde{Z} = \sum_{(\tilde{\sigma}_p^i = \pm 1)} \prod_{\substack{\text{plaquettes} \\ p}} (1 + t\, \tilde{\sigma}_p^1 \tilde{\sigma}_p^2 \tilde{\sigma}_p^3 \tilde{\sigma}_p^4)$$

$$\propto \sum_{\{\tilde{\sigma}_p^i\}} \exp\left[K \sum_p \tilde{\sigma}_p^i \tilde{\sigma}_p^j \tilde{\sigma}_p^k \tilde{\sigma}_p^l \right]$$

$$\boxed{\therefore \; -\beta H_{dual} = K \sum_{\substack{\text{all} \\ \text{plaquettes} \\ p}} \prod \tilde{\sigma}_p^i}$$

Describes a Z_2 Lattice Gauge theory.

Change sign of all Bonds emanating from a site.

$\Downarrow$

each plaquette has 2 such Bonds & hence H is invariant

$\Downarrow$

"Local / Gauge symmetry"

Elitzur's Theorem

↳ spontaneous breaking of local symmetry isn't possible.

∴ while the normal 3d ising shows a p.T., it's dual somehow shouldn't. This is a contradiction!

As a solution to this paradox, Wegner suggested that phase transition occurs without a local order parameter. The two phases are then distinguished based on the asymptotic behaviour of correlation functions.

4. *Ising model in a field:* Consider the partition function for the Ising model ($\sigma_i = \pm 1$) on a square lattice, *in a magnetic field h*; i.e.

$$Z = \sum_{\{\sigma_i\}} \exp\left[K \sum_{\langle ij \rangle} \sigma_i \sigma_j + h \sum_i \sigma_i\right].$$

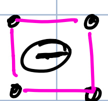
(a) Find the general behavior of the terms in a low-temperature expansion for Z.

(b) Think of a model whose high-temperature series reproduces the generic behavior found in (a); and hence obtain the Hamiltonian, and interactions of the dual model.

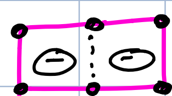
(a) **Low Temperature**

excitations are droplets of $\ominus$ spin islands

Spin flip contribution



$$\longrightarrow \underbrace{e^{-2K\times 4}}_{\#\text{ of nbrs}} \times \underbrace{e^{-2h\times 1}}_{\text{Area}}$$



$$\longrightarrow \underbrace{e^{-2K\times 6}}_{\text{perimeter}} \; \underbrace{e^{-2h\times 2}}_{\text{Area}}$$

$$Z_{\text{low temp}} = Z_0\Big[1 + N e^{-2K\times 4} \underbrace{e^{-2h\times 1}} + 2N e^{-2K\times 6} \underbrace{e^{-2h\times 2}} - - - - - \Big]$$

new terms

ⓑ Model for high Temp dual

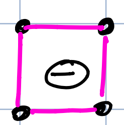
What exactly do we want?

↓

We need to recreate a high temp series that has the form

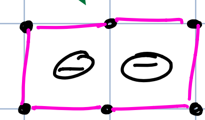
$$Z = (\text{const.}) \sum_{\substack{\text{all}\\ \text{closed}\\ \text{graphs}}} t^{\text{perimeter}}\, b^{\text{area}}$$

, where t & b are some parameters

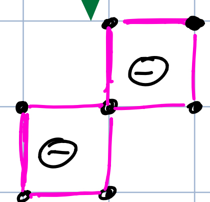


↓

$(t)^4 (b)^1$



$A = 2$
$L = 6$ $\Rightarrow t^6 b^2$



$A = 2$
$L = 8$ $\Rightarrow t^8 b^2$

But how do we do it?

→ 2D duality can't account for area, only perimeter.

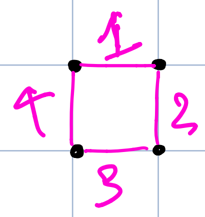
→ Let's start by 1st trying to account for area.

$$\boxed{\text{Our experience in 3d ising hints that an easy way to count area is to use Plaquette terms.}}$$

Try : ① putting ising spins on bonds

$$Z \propto \sum_{\{\sigma_i\}} \prod_{\text{plaq.}} \left(1 + h\, \tilde{\sigma}_p^{\,i}\, \tilde{\sigma}_p^{\,j}\, \tilde{\sigma}_p^{\,k}\, \tilde{\sigma}_p^{\,l}\right)$$

Works?



$$h^{(1)}\ \tilde{\sigma}_p^{\,1}\ \tilde{\sigma}_p^{\,2}\ \tilde{\sigma}_p^{\,3}\ \tilde{\sigma}_p^{\,4}$$

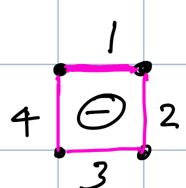
$$\Big\downarrow \sum_{\{\tilde{\sigma}_p\}}$$

$$0 = \underbrace{\left(\sum_{\pm 1} \tilde{\sigma}_p^{\,1}\right)}_{0} \underbrace{\left(\sum_{\pm 1} \tilde{\sigma}_p^{\,2}\right)}_{0} \cdots$$

So vanishes! How to cure this?

→ Simple: for every $\tilde{\sigma}_p^{\,i}$ of a plaquette term, arrange for an additional $\tilde{\sigma}_p^{\,i}$ so that it squares i.e. $\underbrace{(\tilde{\sigma}_p^{\,i})^2 = 1}$ will give contribution.

Therefore,



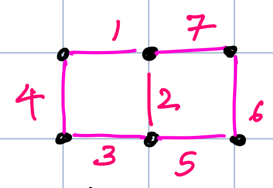
$$\rightarrow h\ \tilde{\sigma}_p^{\,1}\ \tilde{\sigma}_p^{\,2}\ \tilde{\sigma}_p^{\,3}\ \tilde{\sigma}_p^{\,4} \cdot t\tilde{\sigma}_p^{\,1} \cdot t\tilde{\sigma}_p^{\,2} \cdot t\tilde{\sigma}_p^{\,3} \cdot t\tilde{\sigma}_p^{\,4}$$

$$h^{(1)} t^4 \equiv (1, 4)$$

1 → Area

4 → perimeter

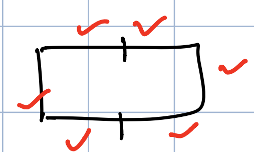
One more



$$h\, \tilde{\sigma}_p^{1}\, \tilde{\sigma}_p^{2}\, \tilde{\sigma}_p^{3}\, \tilde{\sigma}_p^{4} \qquad h\, \tilde{\sigma}_p^{2}\, \tilde{\sigma}_p^{5}\, \tilde{\sigma}_p^{6}\, \tilde{\sigma}_p^{7}$$

$$\left(\tilde{\sigma}_p^{2}\right)^2$$

∴ free $\tilde{\sigma}_p^{i}$ terms $\longrightarrow$ 6



$$\left(t\, \tilde{\sigma}_p^{i}\, \tilde{\sigma}_p^{j}\right)^6 \longrightarrow t^6$$

∴ $h^2 t^6$

$$\therefore\ Z = \sum_{\{\tilde{\sigma}_p\}} \prod_{\text{plaq.}} \left(1 + h\, \tilde{\sigma}_p^{i}\, \tilde{\sigma}_p^{j}\, \tilde{\sigma}_p^{k}\, \tilde{\sigma}_p^{l}\right)\left(1 + t\, \tilde{\sigma}_p^{m}\right)$$

$$\propto\ e^{\ell_1 \sum_{\text{all plaq}} \tilde{\sigma}_p^{i} \tilde{\sigma}_p^{j} \tilde{\sigma}_p^{k} \tilde{\sigma}_p^{l}}\ e^{\ell_2 \sum_{\text{all bonds}} \tilde{\sigma}_p^{m}}$$

$$\therefore\ -\beta H_{dual} = \ell_1 \sum_{\text{all plaq}} \tilde{\sigma}_p^{i} \tilde{\sigma}_p^{j} \tilde{\sigma}_p^{k} \tilde{\sigma}_p^{l} + \ell_2 \sum_{\text{all bonds}} \tilde{\sigma}_p^{m}$$

ℓ_2 → eff magnetic field

Duality Relations :-

$\tilde{h} = e^{-2h}$

$\Rightarrow \tanh l_1 = e^{-2h}$

$t \equiv e^{-2K}$

$\Rightarrow \tanh l_2 = e^{-2K}$

notice the <u>exchange</u> mapping:- $B_1 \longrightarrow J_1$

$B_2 \longrightarrow J_2$